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MODULES SEMICOCRITICAL WITH RESPECT TO A TORSION THEORY AND THEIR APPLICATIONS

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ABSTRACT
The concepts of primitive ideal and semicocritical module with respect to a torsion theory are studied and related to the structure of torsionfree injective modules. Applications are made to the study of (1) composition series with respect to a torsion theory and (2) the structure of endomorphism rings of torsionfree modules. These results are natural generalizations of the properties of certain modules over (noetherian) rings with Krull dimension.

Throughout this paper, R will always denote a ring with identity element, all modules will be unital left R-modules, and τ will be an hereditary torsion theory of modules with torsion class $\mathcal{T}$ and torsionfree class $\mathcal{F}$. Then $\mathcal{L}_\tau = \{I \subseteq R \mid R/I \in \mathcal{T}\}$ will be the filter associated with τ, and $\mathcal{T}(R)$ will denote the torsion submodule of R. A submodule N of a module M is called τ-closed in M if $M/N \in \mathcal{F}$. The τ-closure of a submodule N of M is $\mathrm{Cl}_\tau(N) = \{x \in M \mid \mathrm{ann}_R x \in \mathcal{L}_\tau\}$; hence $\mathrm{Cl}_\tau(N)$ is τ-closed in M. Finally, a module M is τ-cocritical if $M \in \mathcal{F}$ and $M/N \in \mathcal{T}$ for each nonzero submodule N of M. For these definitions and their properties, the reader is referred to [1] or [9].

In [4] Boyle and Feller study modules over a ring with Krull dimension α. In particular, they obtain information about certain modules through the use of an ascending series of modules called the semicritical socle series. Annihilators of α-critical modules are called α-coprimitive ideals and are closely related to the structure of certain injective modules.

Our first two sections are devoted to generalizing the basic theory of [4] from modules over rings with Krull dimension to modules that are torsionfree with

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respect to an hereditary torsion theory τ. In particular, we define τ-semicocritical and τ-primitive ideals based on the corresponding Krull dimension concepts. (Note that we use the dual terminology in order to make our definitions coincide with standard torsion theory terminology.) Our definition of τ-semicocritical also agrees with the one defined by Lau [13] and Golan [11]. We obtain an ascending series from the τ-semicocritical modules analogous to [4] and compare this series to the standard τ-cocritical socle series (e.g., see [6]). The τ-primitive ideals are the annihilators of the τ-cocritical modules; their properties are examined in Section 2.

In the last two sections, we apply our basic results from Sections 1 and 2 to the study of τ-composition series and endomorphism rings. A module $M \in \mathcal{F}$ has a τ-composition series if there exists a chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is τ-cocritical for each $i = 1, 2, \ldots, n$; the least possible such integer n is denoted by $l_\tau(M)$ and is called the τ-length of M, and the nonzero quotients M_i/M_{i-1} are called the factors of the τ-composition series. This type of series has been studied extensively; e.g., see [2], [3], [10], [12], [13], and [17]. Our examination of the factors of a τ-composition series uses both the τ-semicocritical socle series and the theory of τ-primitive ideals. The concept of linkage for ideals and injective modules over a noetherian ring (see [14]) is extended to the torsion theory setting; this linkage concept proves useful both for studying τ-composition series and endomorphism rings of modules in $\mathcal{F}$. We extend some results of Boyle and Feller [5] about endomorphism rings of injective modules over a noetherian ring to the torsion theory case over a general ring. We obtain a bound on the index of nilpotency of ideals that can be used in cases not covered by [11, Proposition 2.3]. Our endomorphism results also relate to some results of [1] and [8].

1. τ-semicocritical modules

A module M will be called τ-semicocritical if there exists a finite set $K_1, K_2, \ldots, K_n$ of submodules of M such that $\bigcap_{i=1}^n K_i = 0$ and M/K_i is τ-cocritical for each $i = 1, 2, \ldots, n$. This concept is closely related to the idea of an α-critical module in the study of Krull dimension (e.g., see [4] and its references) and also agrees with the definition in [11] and [13].

In this section we continue the study of τ-semicocritical modules that was begun in [11] and [13]. After developing the basic properties of τ-semicocritical modules, some of which can be found in [13], we introduce the τ-semicocritical socle and compare it to the commonly studied τ-socle; i.e., the τ-closure of the sum of the τ-cocritical submodules. For later applications, we will be especially concerned with torsion theories for which R has DCC on τ-closed left ideals.

Before stating our first result, we need one more definition. A module M is called τ-full if $M/N \in \mathcal{T}$ whenever N is essential in M. Lau studied this condition in [13], but we will refer to the later paper [11] for the basic properties of τ-full modules due to the relative inaccessability of [13].

PROPOSITION 1.1. *A R-module M is τ-semicocritical if and only if the following conditions hold*:

(1) *M has DCC on τ-closed submodules*;

(2) $M \in \mathcal{F}$;

(3) *M is τ-full.*

PROOF. ($\Leftarrow$) By (1) and (2) M is finite dimensional; hence there exist uniform modules U_i $(i = 1, 2, \ldots, n)$ such that $\bigoplus_{i=1}^n U_i$ is essential in M. For each i, choose submodules K_i of M maximal with respect to $K_i \supseteq \bigoplus_{i \neq j} U_j$ and $K_i \cap U_i = 0$. Then U_i is isomorphic to an essential submodule of M/K_i for each i; so $M/K_i \in \mathcal{F}$ for each i. By [11, Prop. 1.1] each M/K_i is τ-full and hence τ-cocritical. Clearly, $\bigcap_{i=1}^n K_i = 0$.

($\Rightarrow$) Let M/K_i be τ-cocritical, and let $\bigcap_{i=1}^n K_i = 0$. We may assume that n is minimal. Since $\mathcal{F}$ is closed under direct products and submodules, then (2) follows from the exact sequence

$$0 \to M \xrightarrow{\alpha} \prod_{i=1}^{n} M/K_i.$$

By the minimality of n, the restriction of the projection map

$$\prod_{i=1}^{n} M/K_i \to \prod_{i \neq j} M/K_i$$

to $\alpha(M)$ has nonzero kernel. Hence $\alpha(M) \cap (M/K_i) \neq 0$ for each $i \leqq n$. Let N be essential in M. Then $\alpha(N) \cap (M/K_i) \neq 0$ for each i. Since each M/K_i is τ-cocritical, it follows that $\alpha(N)$ is essential in $\Pi_{i=1}^n M/K_i$. Thus $M/N \simeq \alpha(M)/\alpha(N) \subseteq (\Pi_{i=1}^n M/K_i)/\alpha(N) \in \mathcal{T}$ by [10, Prop. 1.2]. Hence M is τ-full; i.e., (3) holds. Finally, since $M \simeq \alpha(M) \in \mathcal{F}$ is contained in $\Pi_{i=1}^n M/K_i$ and since $\Pi_{i=1}^n M/K_i$ has a τ-composition series, then M has a τ-composition series and hence has DCC on τ-closed submodules (see [10], [12] or [17]).

COROLLARY 1.2. *If $M \in \mathcal{F}$ and if M is τ-full, then the following statements are equivalent.*

(1) *M is τ-semicocritical.*

(2) *M has DCC on τ-closed submodules.*
(3) *M is finite dimensional.*

COROLLARY 1.3. *Any submodule of a τ-semicocritical module is τ-semicocritical.*

COROLLARY 1.4. *Let N_1 and N_2 be τ-semicocritical submodules of a module M. Then $N_1 + N_2$ is τ-semicocritical if and only if $N_1 + N_2 \in \mathcal{F}$.*

COROLLARY 1.5. *If N_1 and N_2 are τ-semicocritical submodules of $M \in \mathcal{F}$, then $N_1 + N_2$ is τ-semicocritical.*

The τ-semicocritical modules are closely related to the τ-cocritical modules. In fact, a module is τ-semicocritical if and only if it can be embedded in a direct sum of finitely many τ-cocritical modules. In the remainder of this section, we explore this relationship further.

PROPOSITION 1.6. *A nonzero τ-semicocritical module always contains a nonzero τ-cocritical module.*

PROOF. Let M be a nonzero τ-semicocritical module. Then M is finite dimensional by Corollary 1.2; so M contains a nonzero uniform module U. By Corollary 1.3, U is τ-semicocritical. Since U is also uniform, it must be τ-cocritical.

We define the following two submodules of a module M:

$$\mathrm{S}_\tau(M) = \sum \{N \mid N\ \tau\text{-cocritical}, N \subseteq M\}$$

and

$$\mathrm{Sc}_\tau(M) = \sum \{N \mid N\ \tau\text{-semicocritical}, N \subseteq M\}.$$

The modules $\mathrm{S}_\tau(M)$ and $\mathrm{Sc}_\tau(M)$ are called the τ-cocritical socle and the τ-semicocritical socle, respectively. Since every τ-cocritical module is τ-semicocritical, then $\mathrm{S}_\tau(M) \subseteq \mathrm{Sc}_\tau(M)$. In [4, Example 2.5], Boyle and Feller give an example of a torsion theory for which $\mathrm{S}_\tau(R) \neq \mathrm{Sc}_\tau(R) = R \in \mathcal{F}$; the example arises naturally in the study of Krull dimension over Noetherian rings.

PROPOSITION 1.7. *If M is a module such that* $\mathrm{Sc}_\tau(M) \in \mathcal{F}$, *then* $\mathrm{Sc}_\tau(M) \subseteq \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$ *and hence* $\mathrm{Cl}_\tau(\mathrm{Sc}_\tau(M)) = \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$.

PROOF. If $0 \neq x \in \mathrm{Sc}_\tau(M)$, then x is contained in a sum of finitely many τ-semicocritical modules. Hence Rx is τ-semicocritical by Corollaries 1.4 and 1.3. From Corollaries 1.2 and 1.3 and Proposition 1.6, it follows that Rx contains a finite set of uniform submodules U_i $(i = 1,2,\ldots,n)$ such that $\bigoplus_{i=1}^n U_i$ is essential in Rx. Since Rx is τ-full, then $Rx/(\bigoplus_{i=1}^n U_i) \in \mathcal{T}$. Hence $x \in \mathrm{Cl}_\tau(\bigoplus_{i=1}^n U_i) \subseteq \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$.

Corollary 1.8. *If M is a module with $\mathrm{Sc}_\tau(M) \in \mathcal{F}$, then every nonzero submodule of $\mathrm{Sc}_\tau(M)$ contains a nonzero τ-cocritical module.*

Proof. Let N be a submodule of $\mathrm{Sc}_\tau(M)$, and let $0 \neq x \in N$. As in the proof of Proposition 1.7, Rx is τ-semicocritical. So by Proposition 1.6, there exists a nonzero τ-cocritical module $C \subseteq Rx \subseteq N$.

Proposition 1.9. *Suppose that the filter $\mathcal{L}_\tau$ for τ has a cofinal subset of finitely generated left ideals. If $M \in \mathcal{F}$, then $\mathrm{Sc}_\tau(M) = \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$.*

Proof. From Proposition 1.7, it follows that we only need to show $\mathrm{Cl}_\tau(\mathrm{S}_\tau(M)) \subseteq \mathrm{Sc}_\tau(M)$. Let $0 \neq x \in \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$. Since $M \in \mathcal{F}$, there exists a finitely generated left ideal $I \in \mathcal{L}_\tau$ such that $0 \neq Ix \subseteq \mathrm{S}_\tau(M)$. Thus Ix is finitely generated; so there exists a finite set $C_1, C_2, \ldots, C_n$ of τ-cocritical modules such that $Ix \subseteq \Sigma_{i=1}^n C_i$. By Corollaries 1.4 and 1.3 Ix is τ-semicocritical. So by [11, Prop. 1.1(3)], Rx is τ-full. Since $Rx \subseteq \mathrm{Cl}_\tau(\Sigma_{i=1}^n C_i)$ and since $\Sigma_{i=1}^n C_i$ is a homomorphic image of the module $\bigoplus_{i=1}^n C_i$, which has DCC on τ-closed submodules, then Rx has DCC on τ-closed submodules. Hence Rx is τ-semicocritical by Proposition 1.1.

For a module $M \in \mathcal{F}$, we can form an ascending series of submodules in a canonical manner. Define

$$\mathrm{Sc}_\tau^0(M) = 0, \qquad \mathrm{Sc}_\tau^\alpha(M)/\mathrm{Sc}_\tau^{\alpha-1}(M) = \mathrm{Cl}_\tau(\mathrm{Sc}_\tau(M/\mathrm{Sc}_\tau^{\alpha-1}(M)))$$

when α is not a limit ordinal, and $\mathrm{Sc}_\tau^\alpha(M) = \mathrm{Cl}_\tau(\bigcup_{\beta<\alpha} \mathrm{Sc}_\tau^\beta(M))$ when α is a limit ordinal. We call the chain of modules $\mathrm{Sc}_\tau^\alpha(M)$ the τ-semicocritical socle series for M. There exists a least ordinal λ such that $\mathrm{Sc}_\tau^\lambda(M) = \mathrm{Sc}_\tau^{\lambda+\alpha}(M)$ for all α; λ is called the length of the series.

If R has DCC on τ-closed left ideals, then the filter $\mathcal{L}_\tau$ for τ has a cofinal subset of finitely generated left ideals by the results of [17] and [16]. In this case we have $\mathrm{Sc}_\tau(M) = \mathrm{Cl}_\tau(\mathrm{S}_\tau(M))$ for every $M \in \mathcal{F}$ by Proposition 1.9. Consequently, the τ-semicocritical socle series for $M \in \mathcal{F}$ agrees with a standard τ-cocritical socle series (e.g., see [6]) and $\mathrm{Sc}_\tau^\lambda(M) = M$ for some finite ordinal λ, whenever R has DCC on τ-closed left ideals. We also have $N \cap \mathrm{Sc}_\tau^\alpha(M) = \mathrm{Sc}_\tau^\alpha(N)$ for each submodule N of M and each ordinal α. We shall make use of these facts in Sections 3 and 4.

2. τ-primitive ideals

An ideal D of R will be called τ-primitive if $D = \mathrm{ann}_R C$ for some τ-cocritical module C.

In this section we study the relationships between τ-primitive ideals and τ-semicocritical modules. In case R has DCC on τ-closed left ideals, we show each indecomposable injective module I in $\mathcal{F}$ can be uniquely associated with a minimal τ-primitive ideal $\mathrm{ann}_R S_\tau(I) = \mathrm{ann}_R \mathrm{Sc}_\tau(I)$. Therefore, the minimal τ-primitive ideals can play an important role in studying injective modules in $\mathcal{F}$ and their endomorphism rings.

The results of this section can be viewed as an extension of results on rings with Krull dimension to torsion theories over general rings; see [4] for the Krull dimension case. We also note that our process of associating ideals with indecomposable injectives in $\mathcal{F}$ differs from [1, Section 11] and [2], where prime ideals are associated to injectives in $\mathcal{F}$. In particular, we see in Example 2.6 that a minimal τ-primitive ideal need not be prime.

We begin with an elementary lemma.

LEMMA 2.1. *The annihilator of any nonzero τ-semicocritical module is a finite intersection of τ-primitive ideals.*

PROOF. If N is τ-semicocritical and nonzero, then $N \subseteq \bigoplus_{i=1}^n C_i$, where each C_i is τ-cocritical. Let $p_j : \bigoplus_{i=1}^n C_i \to C_j$ be the projection to the j-th coordinate. Since each $p_j(N) \subseteq C_j$, then each $p_j(N)$ is τ-cocritical, and hence $\mathrm{ann}_R p_j(N)$ is τ-primitive. Since $\mathrm{ann}_R N = \bigcap_{j=1}^n \mathrm{ann}_R p_j(N)$, we are done.

Now we relate the τ-primitive ideals to the annihilators of τ-semicocritical socles.

THEOREM 2.2. *Let R have DCC on τ-closed two-sided ideals of R, and let M be an R-module with $\mathrm{Sc}_\tau(M) \in \mathcal{F}$. Then there exists a τ-semicocritical submodule N of M such that*

$$\mathrm{ann}_R \mathrm{Sc}_\tau(M) = \mathrm{ann}_R N = B_1 \cap B_2 \cap \cdots \cap B_k$$

where each B_i is a τ-primitive ideal.

PROOF. We may assume that $M \neq 0$; so by Corollary 1.8, there exists a nonzero τ-cocritical module N_1 of M. Let $D_1 = \mathrm{ann}_R N_1$. If $D_1(\mathrm{Sc}_\tau(M)) \neq 0$, then there exists a τ-cocritical submodule N_2 of M such that $D_1 N_2 \neq 0$. Let $D_2 = \mathrm{ann}_R(N_1 + N_2)$. Then $D_1 \supsetneqq D_2$. If $D_2(\mathrm{Sc}_\tau(M)) \neq 0$, we continue in this manner to find τ-cocritical modules N_i and annihilators $D_i =$

$\mathrm{ann}_R(N_1 + N_2 + \cdots + N_i)$ such that $D_1 \supset D_2 \supset D_3 \supset \cdots$. Since each D_i is a τ-closed two-sided ideal, then this descending chain must terminate. Hence there is a least integer m such that $D_m(\mathrm{Sc}_\tau(M)) = 0$ and $D_m = \mathrm{ann}_R(N_1 + \cdots + N_m)$. From Corollary 1.4, it follows that $N = N_1 + N_2 + \cdots + N_m$ is τ-semicocritical. Hence

$$\mathrm{ann}_R(\mathrm{Sc}_\tau(M)) = \mathrm{ann}_R(N) = D_m$$

is a finite intersection of τ-primitive ideals by Lemma 2.1.

We note that R having the DCC on τ-closed two-sided ideals does not force $\mathrm{Sc}_\tau(M)$ to be a Δ-module (i.e., a module with DCC on annihilators of subsets) in Theorem 2.2. The condition that R has DCC on τ-closed left ideals does force modules in $\mathscr{F}$ to be Δ-modules. The concept of a Δ-module has received considerable study; e.g., see [1], [5], or [8]. We pursue this idea further in our next few results.

LEMMA 2.3. *Let I be τ-primitive in R. If R has DCC on τ-closed left ideals, then R/I is τ-semicocritical.*

PROOF. Let C be a τ-cocritical module such that $I = \mathrm{ann}_R C$. Thus $0 = \bigcap_{x \in C} \mathrm{ann}_{R/I} x$. Also $(R/I)/\mathrm{ann}_{R/I} x \simeq R/\mathrm{ann}_R x \simeq Rx \subseteq C \in \mathscr{F}$; so $\mathrm{ann}_{R/I} x$ is τ-closed in R/I and $(R/I)/\mathrm{ann}_{R/I} x$ is τ-cocritical for each $x \in C$. Since R has DCC on τ-closed submodules and since the intersection of τ-closed submodules is τ-closed, it follows that there exists a finite set of elements $x_1, x_2, \ldots, x_n$ of C such that $0 = \bigcap_{i=1}^n \mathrm{ann}_{R/I} x_i$. Hence R/I is τ-semicocritical.

LEMMA 2.4. *Let R have DCC on τ-closed left ideals, let C be a nonzero τ-cocritical module, and let $D = \mathrm{ann}_R C$. If C' is a nonzero τ-cocritical module and $DC' = 0$, then $E(C) \simeq E(C')$.*

PROOF. Let $D' = \mathrm{ann}_R C'$. By Lemma 2.3, both R/D and R/D' are τ-semicocritical. For some m and n, $R/D \subseteq \bigoplus_{i=1}^m C$ and $R/D' \subseteq \bigoplus_{i=1}^n C'$. Since $DC' = 0$ by hypothesis, then $D \subseteq D'$; so there exists a natural homomorphism $f: R/D \to R/D'$. Since R/D is τ-full and $C' \in \mathscr{F}$, there exists a nonzero submodule $X \subseteq R/D$ such that $X \cap \ker f = 0$. Let $p: R/D \to C$ be a projection such that $p(X) \neq 0$. Since R/D is τ-full, so is X [11, Prop. 1.1(1)]. Hence there exists a nonzero $W \subseteq X$ such that $W \cap \ker p = 0$ (as $C \in \mathscr{F}$) and $f(W) \neq 0$. Since R/D' is τ-full, so is $f(W)$. Choose a projection $p': R/D' \to C'$ such that $p'f(W) \neq 0$. Since $f(W)$ is τ-full and $C' \in \mathscr{F}$, there exists a nonzero submodule V of $f(W)$ such that $V \cap \ker p' = 0$. Thus we have $p'(V) \subseteq C'$ and

$$0 \neq p'(V) \simeq f^{-1}(V) \cap W \simeq p(f^{-1}(V) \cap W) \subseteq C.$$

Since C and C' are τ-cocritical, they are uniform. But uniform modules with isomorphic nonzero submodules have isomorphic injective hulls; i.e.,

$$E(C) \simeq E(p(f^{-1}(V) \cap W)) \simeq E(p'(V)) \simeq E(C').$$

PROPOSITION 2.5. *Let R have DCC on τ-closed left ideals. If I is an indecomposable injective module with $I \in \mathcal{F}$ and if D is a minimal τ-primitive ideal such that* $\operatorname{ann}_I D \neq 0$, *then* $D = \operatorname{ann}_R \mathrm{S}_\tau(I)$.

PROOF. Let $0 \neq x \in \operatorname{ann}_I D$. Then $Rx \in \mathcal{F}$ and Rx is a uniform R/D-module. Since R/D is τ-full by Lemma 2.3 and Proposition 1.1, then Rx contains a copy of a left ideal K of R/D. By Corollary 1.3, K is τ-semicocritical. But since K is uniform, then K must in fact be τ-cocritical. By Lemma 2.4, we have $E(C) \simeq E(K) \simeq I$ for a τ-cocritical module C with $D = \operatorname{ann}_R C$; we may assume that $C \subseteq I$ by identifying C with its image in I. Thus $\operatorname{ann}_R \mathrm{S}_\tau(I) \subseteq \operatorname{ann}_R C = D$. Hence it is sufficient to show that $\operatorname{ann}_R \mathrm{S}_\tau(I)$ is τ-primitive.

Since I is indecomposable and $I \in \mathcal{F}$, then any τ-semicocritical submodule of I is τ-cocritical; thus $\mathrm{Sc}_\tau(M) = \mathrm{S}_\tau(M)$. It now follows from Theorem 2.2 that $\operatorname{ann}_R \mathrm{Sc}_\tau(M)$ is τ-primitive.

As a consequence of Proposition 2.5, each indecomposable injective module in $\mathcal{F}$ can be uniquely associated with a minimal τ-primitive ideal. We give an elementary example to illustrate this correspondence.

EXAMPLE 2.6. Let $\mathcal{R}$ denote the real numbers, and let $\mathcal{Q}$ be the rational numbers. Then let

$$R = \begin{bmatrix} \mathcal{Q} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & 0 & \mathcal{Q} & \mathcal{R} \\ 0 & 0 & 0 & \mathcal{R} \end{bmatrix}$$

Let e_{ij} denote the matrix with 1 in the i,j-position and 0 elsewhere. Then R has four maximal (left) ideals: $M_1 = R(1 - e_{11})$, $M_3 = R(1 - e_{33}) + Re_{23}$,

$$M_2 = \begin{bmatrix} \mathcal{Q} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & 0 & \mathcal{R} & \mathcal{R} \\ 0 & 0 & \mathcal{Q} & \mathcal{R} \\ 0 & 0 & 0 & \mathcal{R} \end{bmatrix} \quad \text{and} \quad M_4 = \begin{bmatrix} \mathcal{Q} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & 0 & \mathcal{Q} & \mathcal{R} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\mathcal{T}$ be the torsion class generated by the simple modules R/M_1 and R/M_3. Since R is left semiartinian, then the corresponding torsionfree class $\mathcal{F}$ has precisely two nonisomorphic indecomposable injective modules: $E(R/M_2) \simeq Re_{44}/(Re_{44} \cap \mathrm{Soc}\, R)$, where $\mathrm{Soc}\, R$ denotes the left socle of R, and $E(R/M_4) \simeq R/M_4$. Now $S_\tau(E(R/M_2)) \simeq (Re_{44} \cap M_4)/(Re_{44} \cap \mathrm{Soc}\, R)$ has corresponding minimal τ-primitive ideal

$$\mathrm{ann}_R S_\tau(E(R/M_2)) = Re_{44} + \mathrm{Soc}\, R = \begin{bmatrix} \mathcal{Q} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ 0 & 0 & 0 & \mathcal{R} \\ 0 & 0 & 0 & \mathcal{R} \\ 0 & 0 & 0 & \mathcal{R} \end{bmatrix},$$

and $S_\tau(E(R/M_4)) \simeq R/M_4$ has corresponding minimal τ-primitive ideal M_4. We also point out that R has DCC on τ-closed left ideals since $R/\mathcal{T}(R)$ has a $\mathcal{T}$-composition series with four nonzero factors that are isomorphic to R/M_2, $Re_{33}/(Re_{33} \cap \mathrm{Soc}\, R)$, $(Re_{44} \cap M_4)/(Re_{44} \cap \mathrm{Soc}\, R)$, and R/M_4.

We end the section with two elementary lemmas on annihilators that will be useful in the last two sections.

LEMMA 2.7. *Let* $M = \bigoplus_{j \in A} M_j$ *be a module with* $\mathrm{Sc}_\tau(M) \in \mathcal{F}$, *and let* $D_1, D_2, \ldots, D_n$ *be* τ*-primitive ideals.*

(1) *If* $(D_1 \cap D_2 \cap \cdots \cap D_n)(\mathrm{Sc}_\tau(M)) = 0$, *then for each* $j \in A$ *there exists* D_i *such that* $\mathrm{ann}_{M_j} D_i \neq 0$ *or else* $\mathrm{Sc}_\tau(M_j) = 0$.

(2) *If* $R \neq \mathrm{ann}_R \mathrm{Sc}_\tau(M) = D_1 \cap D_2 \cap \cdots \cap D_n$ *and if this intersection is irredundant, then for each* D_i *there exists* $j \in A$ *such that* $\mathrm{ann}_{M_j} D_i \neq 0$.

PROOF. (1) Let $j \in A$. Suppose that $\mathrm{Sc}_\tau(M_j) \neq 0$. Since $\mathrm{Sc}_\tau(M_j) \subseteq \mathrm{Sc}_\tau(M)$, then

$$(D_1 D_2 \cdots D_n)(\mathrm{Sc}_\tau(M_j)) \subseteq (D_1 \cap D_2 \cap \cdots \cap D_n)(\mathrm{Sc}_\tau(M)) = 0$$

by hypothesis. If $(D_2 D_3 \cdots D_n)(\mathrm{Sc}_\tau(M_j)) \neq 0$, then $\mathrm{ann}_{M_j} D_1 \neq 0$. Otherwise, $(D_2 D_3 \cdots D_n)(\mathrm{Sc}_\tau(M_j)) = 0$. Continuing in this manner, we eventually find D_i such that $\mathrm{ann}_{M_j} D_i \neq 0$.

(2) Suppose that there exists D_i such that $\mathrm{ann}_{M_j} D_i = 0$ for all $j \in A$. Then $\mathrm{ann}_M D_i = 0$. Since

$$D_i(D_1 \cap D_2 \cap \cdots \cap D_{i-1} \cap D_{i+1} \cap \cdots \cap D_n)(\mathrm{Sc}_\tau(M)) = 0,$$

then

$$(D_1 \cap D_2 \cap \cdots \cap D_{i-1} \cap D_{i+1} \cap \cdots \cap D_n)(\mathrm{Sc}_\tau(M)) = 0.$$

This contradicts the irredundancy of $D_1 \cap \cdots \cap D_n$.

LEMMA 2.8. *Let R have DCC on τ-closed left ideals, and let M be a module in $\mathcal{F}$. If $D_1, D_2, \ldots, D_n$ are τ-primitive ideals such that*

$$(D_1 \cap D_2 \cap \cdots \cap D_n)(\mathrm{Sc}_\tau(M)) = 0,$$

then $\mathrm{ann}_M(D_1 \cap D_2 \cap \cdots \cap D_n) = \mathrm{Sc}_\tau(M)$.

PROOF. Let $K = \mathrm{ann}_M(D_1 \cap D_2 \cap \cdots \cap D_n)$. Since each R/D_i is τ-semicocritical by Lemma 2.3, then $R/(D_1 \cap D_2 \cap \cdots \cap D_n)$ must also be τ-semicocritical. So any cyclic submodule of K is τ-semicocritical by [11, Prop. 1.1] and Proposition 1.1. Thus $K \subseteq \mathrm{Sc}_\tau(M)$.

3. Applications to τ-composition series

In this section we apply our previous results to the study of the τ-composition series of a module in $\mathcal{F}$. To aid in this, we use the radical $K_\tau(R)$ introduced in [3], [9], [13] and [15]. The relationships of the τ-composition series, the τ-semicocritical socle series, and the τ-primitive ideals are discussed. If R has DCC on τ-closed left ideals, then the relationship of $\mathrm{S}_\tau(I)$ and the indecomposable injective $I \in \mathcal{F}$ are examined in Theorem 3.8.

The τ-radical of R, denoted by $K_\tau(R)$, is the intersection of all τ-primitive ideals of R (or $K_\tau(R) = R$ if there are no proper τ-primitive ideals). The τ-radical and its basic properties have been discussed in [3], [9], [13], and [15]. Every τ-cocritical module is annihilated by $K_\tau(R)$, $R/K_\tau(R) \in \mathcal{F}$, and $\mathcal{T}(R) \subseteq K_\tau(R)$. If R has DCC on τ-closed two-sided ideals, then $K_\tau(R)$ is a finite intersection of minimal τ-primitive ideals.

LEMMA 3.1. *Let $K_\tau(R)/\mathcal{T}(R)$ have a τ-composition series with τ-length n. If $M \in \mathcal{F}$, then there exists a least integer m such that $(K_\tau(R))^m M = 0$.*

PROOF. Since $K_\tau(R)/\mathcal{T}(R)$ has a τ-composition series of τ-length n and since $K_\tau(R)$ annihilates every τ-critical module, then $(K_\tau(R))^{n+1} \subseteq \mathcal{T}(R)$. Hence $(K_\tau(R))^{n+1} M \subseteq \mathcal{T}(R)M = 0$, as $M \in \mathcal{F}$. Thus a minimal integer m exists.

We can now give a result that relates the τ-semicocritical socle series of a module $M \in \mathcal{F}$ to the τ-composition series for $K_\tau(R)/\mathcal{T}(R)$.

THEOREM 3.2. *Let R have DCC on τ-closed left ideals, and let $M \in \mathcal{F}$. Then $K_\tau(R)/\mathcal{T}(R)$ has a τ-composition series, and there exists a least integer $m \leqq l_\tau(K_\tau(R)/\mathcal{T}(R)) + 1$ such that $(K_\tau(R))^m M = 0$ and $\mathrm{Sc}_\tau^i(M) = \mathrm{ann}_M(K_\tau(R))^i$ for each $i = 1, 2, \ldots, m$.*

PROOF. Since R has DCC on τ-closed left ideals, then by [17, Theorem 1.4] R also has ACC on τ-closed left ideals. Hence $R/\mathcal{T}(R)$ has a τ-composition series by [12, Prop. 1.2]; thus $K_\tau(R)/\mathcal{T}(R)$ also has a τ-composition series. By Lemma 3.1 there exists a least integer m such that $(K_\tau(R))^m M = 0$. It remains to show that $\mathrm{Sc}_\tau^i(M) = \mathrm{ann}_M(K_\tau(R))^i$ for each $i \leqq m$.

By Lemma 2.8, $\mathrm{ann}_M(K_\tau(R)) = \mathrm{Sc}_\tau(M)$. From the remarks following Proposition 1.9, we see that $M/\mathrm{Sc}_\tau(M) \in \mathcal{F}$. By Lemma 2.8,

$$\mathrm{ann}_{M/\mathrm{Sc}_\tau(M)}(K_\tau(R)) = \mathrm{Sc}_\tau(M/\mathrm{Sc}_\tau(M)) = \mathrm{Sc}_\tau^2(M)/\mathrm{Sc}_\tau(M).$$

Thus $\mathrm{ann}_M(K_\tau(R))^2 = \mathrm{Sc}_\tau^2(M)$. Proceeding recursively, we see that $\mathrm{ann}_M(K_\tau(R))^i = \mathrm{Sc}_\tau^i(M)$ for each $i \leqq m$.

PROPOSITION 3.3. *If R has DCC on τ-closed left ideals, then R has only finitely many minimal τ-primitive ideals.*

PROOF. We may assume that τ is nontrivial. Since R has DCC on τ-closed left ideals, it follows that R has at least one minimal τ-primitive ideal and that $K_\tau(R) = D_1 \cap D_2 \cap \cdots \cap D_n$ for some finite set of minimal τ-primitive ideals.

Now let D be any minimal τ-primitive ideal of R, and let $D = \mathrm{ann}_R C$, where C is τ-cocritical. Then $D \supseteq K_\tau(R)$, so that $(D_1 \cap D_2 \cap \cdots \cap D_n)C = 0$. By Lemma 2.7, $\mathrm{ann}_C D_i \neq 0$ for some i, and hence $\mathrm{ann}_{E(C)} D_i \neq 0$. Thus $D_i = \mathrm{ann}_R S_\tau(E(C))$ by Proposition 2.5. In particular, $D_i \subseteq \mathrm{ann}_R C = D$. Since D is a minimal τ-primitive ideal, we must have $D_i = D$.

REMARK 3.4. If R has DCC on τ-closed left ideals, then $R/K_\tau(R)$ has a τ-composition series and R has finitely many minimal τ-primitive ideals $D_1, D_2, \ldots, D_n$ by Proposition 3.3. By using Proposition 2.5 and results of Goldman [12], it is possible to show that $l_\tau(R/K_\tau(R)) = \sum_{i=1}^n l_\tau(R/D_i)$.

REMARK 3.5. In view of Propositions 3.3 and 2.5 and Lemma 2.4, we see that if R has DCC on τ-closed left ideals, then there are only finitely many nonisomorphic indecomposable injective modules in $\mathcal{F}$.

Next we find some relationships between the τ-primitive ideals and modules with a τ-composition series.

PROPOSITION 3.6. *Let D be a τ-primitive ideal of R, and let $M \in \mathcal{F}$ be a module with a τ-composition series. If D annihilates a nonzero τ-cocritical factor in a τ-composition series for M, then D annihilates a nonzero submodule of $\mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M)$ for some i. If R has DCC on τ-closed left ideals and if D is a minimal τ-primitive ideal of R, then the converse holds.*

PROOF. Suppose that D annihilates a factor C in a τ-composition series for M. Since

$$0 \subset \mathrm{Sc}_\tau(M) \subset \mathrm{Sc}_\tau^2(M) \subset \cdots \subset \mathrm{Sc}_\tau^n(M) = M$$

is a chain of τ-closed submodules of M, then the chain can be refined to a τ-composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M.$$

By results of [12], $E(C) \simeq E(M_i/M_{i-1})$ for some i. Let t be the largest integer such that $\mathrm{Sc}_\tau^t(M) \subseteq M_{i-1}$. Then $M_i \subseteq \mathrm{Sc}_\tau^{t+1}(M)$. Choose $x \in M_i - M_{i-1}$. From Corollaries 1.4 and 1.3, it follows that $A = (Rx + \mathrm{Sc}_\tau^t(M))/\mathrm{Sc}_\tau^t(M)$ is τ-semicocritical. Let $B = ((Rx \cap M_{i-1}) + \mathrm{Sc}_\tau^t(M))/\mathrm{Sc}_\tau^t(M)$. Then

$$\begin{aligned} A/B &\simeq (Rx + \mathrm{Sc}_\tau^t(M))/((Rx \cap M_{i-1}) + \mathrm{Sc}_\tau^t(M)) \\ &\simeq (Rx + \mathrm{Sc}_\tau^t(M))/((Rx + \mathrm{Sc}_\tau^t(M)) \cap M_{i-1}) \\ &\simeq (Rx + M_{i-1})/M_{i-1} \subseteq M_i/M_{i-1} \in \mathcal{F}. \end{aligned}$$

Since A is τ-semicocritical (and hence τ-full), then B is not essential in A. Hence A has a nonzero submodule C' such that C' is isomorphic to a submodule of M_i/M_{i-1}. Let C'' be the nonzero submodule of C' that is carried into C under the mappings $C' \to E(M_i/M_{i-1}) \simeq E(C)$. Then D annihilates C'' and $C'' \subseteq \mathrm{Sc}_\tau^{t+1}(M)/\mathrm{Sc}_\tau^t(M)$ as desired.

Conversely, assume that D is a minimal τ-primitive ideal and that D annihilates a nonzero submodule C of $\mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M)$. We may assume that C is τ-cocritical by Corollary 1.8. Let $N/\mathrm{Sc}_\tau^{i-1}(M) = \mathrm{Cl}_\tau(C)$ in $M/\mathrm{Sc}_\tau^{i-1}(M)$. Then

$$0 \subseteq \mathrm{Sc}_\tau^{i-1}(M) \subset N \subseteq \mathrm{Sc}_\tau^i(M) \subseteq M$$

can be refined to a τ-composition series of M, and $N/\mathrm{Sc}_\tau^{i-1}(M)$ is a factor in this τ-composition series. Then $C \subseteq N/\mathrm{Sc}_\tau^{i-1}(M) \subseteq E(C)$; so $D = \mathrm{ann}_R S_\tau(E(C))$ by Proposition 2.5. Hence D annihilates the factor $N/\mathrm{Sc}_\tau^{i-1}(M)$ as desired.

Let R have DCC on τ-closed left ideals, and let $0 \neq M \in \mathcal{F}$. By Theorem 3.2, there exists a finite τ-semicocritical socle series of M:

$$M = \mathrm{Sc}_\tau^n(M) \supset \mathrm{Sc}_\tau^{n-1}(M) \supset \cdots \supset \mathrm{Sc}_\tau(M) \supset 0.$$

If D is a τ-primitive ideal, then we say that D is linked to M at the i-th layer of M if D annihilates some nonzero submodule of $\mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M)$, where $1 \leqq i \leqq n$. Then we say that D is linked to M if D is linked to the i-th layer of M

for some i. This definition is motivated by the corresponding definition of a prime ideal linked to an injective module over a noetherian ring [14] and of an α-coprimitive ideal linked to a module over a ring with Krull dimension α [5].

THEOREM 3.7. *Let R have DCC on τ-closed left ideals, let D be a τ-primitive ideal of R, and let $M \in \mathcal{F}$. Then D annihilates a nonzero factor in a τ-composition series of a cyclic submodule of M if and only if D is linked to M.*

PROOF. If D is linked to M, then there exists $i \geqq 1$ and a cyclic module $Rx \subseteq \mathrm{Sc}_\tau^i(M)$ such that $Rx \not\subseteq \mathrm{Sc}_\tau^{i-1}(M)$ and D annihilates $(Rx + \mathrm{Sc}_\tau^{i-1}(M))/\mathrm{Sc}_\tau^{i-1}(M)$. Since R has DCC on τ-closed left ideals, it follows from [17, Theorem 1.4] and results of Goldman [12] that Rx has a τ-composition series. Note that $Rx = Rx \cap \mathrm{Sc}_\tau^i(M)$ and $\mathrm{Sc}_\tau^{i-1}(Rx) = Rx \cap \mathrm{Sc}_\tau^{i-1}(M)$, so that $0 \neq Rx/\mathrm{Sc}_\tau^{i-1}(Rx) \in \mathcal{F}$ and $D(Rx/\mathrm{Sc}_\tau^{i-1}(Rx)) = 0$. Since Rx has a τ-composition series, then any chain of τ-closed submodules can be extended to a τ-composition series; so D annihilates all the factors of the τ-composition series that are between $\mathrm{Sc}_\tau^{i-1}(Rx)$ and Rx.

Conversely, assume that D annihilates a factor in a τ-composition series of some cyclic module $Rx \subseteq M$. (Note that every cyclic module in $\mathcal{F}$ has a τ-composition series by the results of [17] and [12].) Applying Proposition 3.6 to Rx, we see that D annihilates a nonzero submodule A of $\mathrm{Sc}_\tau^i(Rx)/\mathrm{Sc}_\tau^{i-1}(Rx)$ for some i. Consider the natural homomorphism

$$f : \mathrm{Sc}_\tau^i(Rx)/\mathrm{Sc}_\tau^{i-1}(Rx) \to \mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M).$$

Since

$$\mathrm{Sc}_\tau^i(Rx) \cap \mathrm{Sc}_\tau^{i-1}(M) = (Rx \cap \mathrm{Sc}_\tau^i(M)) \cap \mathrm{Sc}_\tau^{i-1}(M) = Rx \cap \mathrm{Sc}_\tau^{i-1}(M) = \mathrm{Sc}_\tau^{i-1}(Rx),$$

then f is $1-1$. Hence $\mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M)$ contains a copy of A, and thus D is linked to M.

We shall pursue the idea of linkage further in our study of endomorphisms. But next we wish to examine the size of the τ-critical socle of an indecomposable injective module. We recall from Proposition 2.5 that each indecomposable injective module I in $\mathcal{F}$ can be uniquely associated with a minimal τ-primitive ideal $D = \mathrm{ann}_R \mathrm{S}_\tau(I)$, provided that R has DCC on τ-closed left ideals.

THEOREM 3.8. *Let R have DCC on τ-closed left ideals, and let I be an indecomposable injective module in $\mathcal{F}$ with associated minimal τ-primitive ideal D. Then $\mathrm{S}_\tau(I) \neq I$ if and only if D annihilates a nonzero τ-cocritical factor in a τ-composition series of $K_\tau(R)/\mathcal{T}(R)$. Consequently, $\mathrm{S}_\tau(I) \neq I$ if and only if*

$I \simeq E(C)$, where C is a nonzero τ-cocritical factor in some τ-composition series for $K_\tau(R)/\mathcal{T}(R)$.

PROOF. First, suppose that $S_\tau(I) \neq I$. If $K_\tau(R)I = 0$, then by Lemma 2.8, we have $I = \mathrm{ann}_I K_\tau(R) = \mathrm{Sc}_\tau(I) = S_\tau(I)$, which is a contradiction. Therefore, $K_\tau(R)I \neq 0$. Since R has DCC on τ-closed left ideals, then $K_\tau(R)/\mathcal{T}(R)$ has a τ-composition series (which is non-trivial as $\mathcal{T}(R)I = 0$). Let

$$K_\tau(R) = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \mathcal{T}(R)$$

such that each M_i/M_{i-1} is τ-cocritical, and let D_i be the minimal τ-primitive ideal associated with $E(M_i/M_{i-1})$ for each i $(1 \leqq i \leqq k)$. Now $D_1D_2D_3 \cdots D_kK_\tau(R)I \subseteq \mathcal{T}(R)I = 0$, and thus $\mathrm{ann}_I D_j \neq 0$ for some $j \leqq k$. By Proposition 2.5, $D_j = \mathrm{ann}_R(S_\tau(I)) = D$, and thus D annihilates M_j/M_{j-1}.

Conversely, suppose that D annihilates a nonzero τ-cocritical factor in a τ-composition series for $K_\tau(R)/\mathcal{T}(R)$. Let $S_i/\mathcal{T}(R) = \mathrm{Sc}^i_\tau(K_\tau(R)/\mathcal{T}(R))$ for each $i \geqq 0$. By Proposition 3.6 D annihilates a nonzero submodule $C = B/S_{i-1}$ of some S_i/S_{i-1}. Without loss of generality, we may assume that C is τ-cocritical. By Lemma 2.4, $E(C) \simeq I$, and thus it suffices to show that $S_\tau(E(C)) \neq E(C)$.

In order to obtain a contradiction, we assume that $S_\tau(E(C)) = E(C)$. Then $K_\tau(R)E(C) = 0$. Thus $E(C)$ is a R/S_{i-1}-module and is the injective hull of C as an R/S_{i-1}-module. Since $B \subseteq K_\tau(R)$, we also have $CE(C) = 0$. If $J \in \mathcal{F}$ is indecomposable and injective as an R/S_{i-1}-module, we claim that $CJ = 0$. For if not, choose $x \in J$ such that $Cx \neq 0$. Since C is τ-cocritical and $J \in \mathcal{F}$, the mapping $C \to Cx$ via $c \to cx$ is a monomorphism; so $J \simeq E(C)$, which is a contradiction. Therefore, $CJ = 0$, as we claimed. Thus C annihilates every indecomposable injective R/S_{i-1}-module that is in $\mathcal{F}$ as an R-module. Since $E(R/S_{i-1})$ is a direct sum of indecomposable injective modules in $\mathcal{F}$ by [17, Theorem 1.4] and [16, Theorem 1.2], then C annihilates $E(R/S_{i-1})$, which contradicts $C \neq 0$.

The last statement of the theorem follows from the first part of the theorem, Proposition 2.5, and Lemma 2.4.

If R has DCC on τ-closed left ideals, then the following corollary shows that the torsion theory is closely related to Goldie's torsion theory whenever $\mathrm{Sc}_\tau(I) = I$ for each indecomposable injective module I in $\mathcal{F}$.

COROLLARY 3.9. *Let R have DCC on τ-closed left ideals, and let $J_1, J_2, \ldots, J_n$ be a complete set of representatives of the nonisomorphic indecomposable injective modules in $\mathcal{F}$. (See Remark 3.5.) Then the following statements are equivalent.*

(1) $S_\tau(J_i) = J_i$ *for all* $i = 1, 2, \ldots, n$.

(2) $K_\tau(R) = \mathcal{T}(R)$.

(3) $R/\mathcal{T}(R)$ *is* τ-*full.*

(4) $\mathrm{Sc}_\tau(M) = M$ *for all* $M \in \mathcal{F}$.

PROOF. (1) $\Leftrightarrow$ (2). This is immediate from Theorem 3.8.

(2) $\Rightarrow$ (3). This follows from Proposition 1.1 and the fact that $R/K_\tau(R)$ is τ-semicocritical.

(3) $\Rightarrow$ (4). Every cyclic module in $\mathcal{F}$ is a homomorphic image of $R/\mathcal{T}(R)$. Thus it follows from (3), [11, Proposition 1.1], and Proposition 1.1 that every cyclic module in $\mathcal{F}$ is τ-semicocritical. Hence $\mathrm{Sc}_\tau(M) = M$ for all $M \in \mathcal{F}$.

(4) $\Rightarrow$ (1). Since each J_i is uniform, then $\mathrm{Sc}_\tau(J_i) = S_\tau(J_i)$ for each i; so (4) $\Rightarrow$ (1) is trivial.

4. Applications to endomorphism rings

In this section, we apply our previous results to study the endomorphism rings of torsionfree modules over a ring with DCC on τ-closed left ideals. Our main results use the concept of linkage, as defined in Section 3, to study the endomorphism rings of injective modules. We are aided in this study by the fact that, for a ring with DCC on τ-closed left ideals, there are only finitely many nonisomorphic indecomposable injective modules in $\mathcal{F}$. (See Remark 3.5.) Our results extend from [5] the case of endomorphism rings of injective modules with Krull dimension over noetherian rings to the relative case over general rings. We also obtain a result (Theorem 4.8) that provides new information about the index of nilpotency of ideals of endomorphism rings as studied in Section 2 of [11].

We begin our study by writing $\mathrm{Sc}_\tau(M)$ in terms of homomorphisms.

PROPOSITION 4.1. *Let R have DCC on τ-closed left ideals, and let $J_1, J_2, \ldots, J_n$ be a complete set of representatives of the nonisomorphic indecomposable injective modules in $\mathcal{F}$. If $M \in \mathcal{F}$, then*

$$\mathrm{Sc}_\tau(M) = \cap \left\{ \ker f \,\middle|\, f \in \bigcup_{i=1}^{n} \mathrm{Hom}_R(M, J_i),\ \ker f \textit{ essential in } M \right\}.$$

PROOF. Let $H = \{f \mid f \in \bigcup_{i=1}^{n} \mathrm{Hom}_R(M, J_i),\ \ker f$ essential in $M\}$. If $f \in H$ and N is a τ-semicocritical submodule of M, then $N \cap \ker f$ is essential in N, and hence $f(N) \in \mathcal{T}$ by Proposition 1.1. Since $f(N) \subseteq J_i \in \mathcal{F}$ for some i, we have $f(N) \in \mathcal{T} \cap \mathcal{F} = 0$; so $N \subseteq \ker f$. Therefore, $\mathrm{Sc}_\tau(M) \subseteq \cap \{\ker f \mid f \in H\} = W$.

Write $E(M/\mathrm{Sc}_\tau(M)) \simeq \bigoplus_{j \in A} I_j$, where each I_j is indecomposable and in $\mathcal{F}$ (see [17, Theorem 1.4] and [16, Theorem 1.2]). For each $j \in A$, let p_j be the projection to the j-th coordinate of this direct sum. Since R has DCC on τ-closed left ideals, then $\mathrm{Sc}_\tau(M)$ is essential in M. Consequently, the composition f_i

$$M \to M/\mathrm{Sc}_\tau(M) \to E(M/\mathrm{Sc}_\tau(M)) \simeq \bigoplus_{j \in A} I_j \xrightarrow{p_i} I_i$$

has essential kernel. Since each $I_i \simeq J_k$ for some k, then $W \subseteq \ker f_i$ for each $i \in A$. Hence the image of W under the composition

$$M \to M/\mathrm{Sc}_\tau(M) \to E(M/\mathrm{Sc}_\tau(M)) \simeq \bigoplus_{j \in A} I_j$$

is 0. Thus $W \subseteq \mathrm{Sc}_\tau(M)$.

Therefore, $W = \mathrm{Sc}_\tau(M)$ as desired.

The following consequence of Proposition 4.1 can be contrasted with Theorem 3.8.

COROLLARY 4.2. *Let R have DCC on τ-closed left ideals, and let $J_1, J_2, \ldots, J_n$ be a complete set of representatives of the nonisomorphic indecomposable injective modules in $\mathcal{F}$. Then the following statements are equivalent for a nonzero indecomposable injective module I in $\mathcal{F}$.*

(1) $\mathrm{S}_\tau(I) = I$.

(2) *For each $k \leqq n$, every nonzero homomorphism from I into J_k is an isomorphism.*

(3) $\mathrm{Hom}_R(I, I)$ *is a division ring, and* $\mathrm{Hom}_R(I, J_k) = 0$ *whenever $J_k \not\simeq I$.*

PROOF. (1) $\Rightarrow$ (2). If $f \in \mathrm{Hom}_R(I, J_k)$ and $\ker f \neq 0$, then $\ker f$ is essential in I. By Proposition 4.1 and (1), we have $0 = f(\mathrm{Sc}_\tau(I)) = f(\mathrm{S}_\tau(I)) = f(I)$, and hence $f = 0$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). It follows from (3) that, for each $k \leqq n$, $f = 0$ is the only map in $\mathrm{Hom}_R(I, J_k)$ with essential kernel. Thus by Proposition 4.1, we have $I = \mathrm{Sc}_\tau(I) = \mathrm{S}_\tau(I)$.

Before giving the main results of this section, we need two more preliminary results that relate the concept of linkage to homomorphisms.

LEMMA 4.3. *Let R have DCC on τ-closed left ideals, let D be a minimal τ-primitive ideal of R, let $I \in \mathcal{F}$ be the indecomposable injective module associated with D, and let $M \in \mathcal{F}$. Then D is linked to the i-th layer of M if and only if there exists a nonzero homomorphism of $\mathrm{Sc}^i_\tau(M)/\mathrm{Sc}^{i-1}_\tau(M)$ into I.*

PROOF. Let $A = \mathrm{Sc}_\tau^i(M)/\mathrm{Sc}_\tau^{i-1}(M)$.

If D is linked to the i-th layer of M, then D annihilates a nonzero submodule C of A. Since R has DCC on τ-closed left ideals, we may assume that C is τ-cocritical. By Lemma 2.4, $E(C) \simeq I$. By the injectivity of I, we can extend the natural embedding $C \to I$ to a homomorphism $A \to I$.

Conversely, suppose that $f: A \to I$ is a nonzero homomorphism. If $\mathrm{S}_\tau(A) \subseteq \ker f$, then $\mathrm{Cl}_\tau(\ker f) = A$; so $f(A) \in \mathcal{T}$. Since $0 \neq f(A) \subseteq I \in \mathcal{F}$, it follows that we cannot have $\mathrm{S}_\tau(A) \subseteq \ker f$. Hence there exists a τ-cocritical submodule C of A such that $C \cap \ker f = 0$, so that $C \simeq f(C) \subseteq \mathrm{Sc}_\tau(I)$. By Proposition 2.5, $D = \mathrm{ann}_R(\mathrm{S}_\tau(I))$. Thus $DC = 0$ as desired.

If D_1 and D_2 are minimal τ-primitive ideals and if I is the indecomposable injective associated with D_2, then we say that D_1 is *linked* to D_2 whenever D_1 is linked to I.

COROLLARY 4.4. *Let R have DCC on τ-closed left ideals. Let D_1 and D_2 be minimal τ-primitive ideals of R with associated indecomposable injective modules I_1 and I_2, respectively. Then D_1 is linked to D_2 if and only if* $\mathrm{Hom}_R(I_2, I_1) \neq 0$.

We can now give our applications of linkage to endomorphism rings of injective modules in $\mathcal{F}$.

Faith [7] gives conditions for $\mathrm{End}_R(I)$ to be a division ring for any injective module I. Our result gives a linkage condition for the endomorphism ring of an indecomposable injective module in $\mathcal{F}$ to be a division ring. Our criterion is analogous to the one obtained by Boyle and Feller [5] in their study of noetherian rings.

THEOREM 4.5. *Let R have DCC on τ-closed left ideals, let $I \in \mathcal{F}$ be an indecomposable injective module, and let D be the minimal τ-primitive ideal associated with I. Then the following statements are equivalent.*

(1) $\mathrm{End}_R(I)$ *is a division ring.*

(2) $\mathrm{Hom}_R(I/K, I) = 0$ *for each nonzero* $K \subseteq I$.

(3) $\mathrm{Hom}_R(I/\mathrm{Sc}_\tau^i(I), I) = 0$ *for each* $i \geqq 1$.

(4) *D is linked to I only at the first layer.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). This follows from Lemma 4.3 and the injectivity of I.

(4) $\Rightarrow$ (1). Let J denote the Jacobson radical of $\mathrm{End}_R(I)$; then $J = \{f \in \mathrm{End}_R(I) \mid \ker f \text{ is essential in } I\}$. Since I is indecomposable, it suffices to show that $J = 0$. If $0 \neq f \in J$, let m be the smallest integer such that $f(\mathrm{Sc}_\tau^m(I)) \neq 0$.

Then f induces a nonzero homomorphism $g : \mathrm{Sc}_\tau^m(I)/\mathrm{Sc}_\tau^{m-1}(I) \to I$. Thus D is linked to I at the m-th layer. By (4), we must have $m = 1$. Hence g is just the restriction of f to $\mathrm{Sc}_\tau(I)$. Since $\ker f$ is essential in I and since $I \in \mathcal{F}$, then $\ker f$ contains $\mathrm{S}_\tau(I) = \mathrm{Sc}_\tau(I)$, which contradicts our choice of M.

THEOREM 4.6. *Let R have DCC on τ-closed left ideals. Let $E = \bigoplus_{j=1}^n I_j$, where each $I_j \in \mathcal{F}$ is an indecomposable injective module ($1 \leqq j \leqq n$). Let D_j be the minimal τ-primitive ideal associated with I_j ($1 \leqq j \leqq n$). Then* $\mathrm{End}_R(E)$ *is semisimple if and only if*

(1) *D_j is linked to I_j only at the first layer ($1 \leqq j \leqq n$), and*

(2) *D_i is not linked to D_j unless $I_i \simeq I_j$.*

PROOF. ($\Rightarrow$) Assume that $\mathrm{End}_R(E)$ is semisimple. Then $J = \{f \in \mathrm{End}_R(E) \mid \ker f \text{ is essential in } E\} = 0$. If D_j is linked to I_j at the i-th layer, then by Proposition 4.3 and the injectivity of I_j there exists a nonzero homomorphism $f : I_j/\mathrm{Sc}_\tau^{i-1}(I_j) \to I_j$. Hence the natural composition

$$I_j \xrightarrow{p} I_j/\mathrm{Sc}_\tau^{i-1}(I_j) \xrightarrow{f} I_j$$

has $\ker fp \supseteq \mathrm{Sc}_\tau^{i-1}(I_j)$. Now fp extends to a map $g : E \to E$ via $g(a_1, a_2, \ldots, a_n) = fp(a_j)$, where $a_t \in I_t$ ($1 \leqq t \leqq n$). Thus

$$\ker g = I_1 \oplus \cdots \oplus I_{j-1} \oplus \ker fp \oplus I_{j+1} \oplus \cdots \oplus I_n.$$

Since $\ker g$ cannot be essential (as $J = 0$), then $\ker fp$ is not essential in I_j. Since $\mathrm{S}_\tau^{i-1}(I_j) \subseteq \ker fp \in \mathcal{F}$, then we must have $i = 1$; i.e., (1) holds.

If $I_t \not\simeq I_j$ and $f : I_t \to I_j$, then $\ker f$ is essential in I_t. Then f extends to a homomorphism $g : E \to E$ via $g(a_1, a_2, \ldots, a_n) = f(a_t)$, where $a_i \in I_i$ ($1 \leqq i \leqq n$). Then $\ker g$ is essential in E. Since $J = 0$, then $g = 0$. Thus by Corollary 4.4, D_t is not linked to D_j.

($\Leftarrow$) By (2) and Corollary 4.4, $\mathrm{Hom}_R(I_i, I_j) = 0$ when $I_i \not\simeq I_j$. By (1) and Theorem 4.5, $\mathrm{End}_R(I_j)$ is a division ring ($1 \leqq j \leqq n$). Hence $\mathrm{End}_R(E)$ is semisimple.

COROLLARY 4.7. *Let R have DCC on τ-closed left ideals, and let $I_1, I_2, \ldots, I_n$ be a complete set of representatives of the isomorphism classes of indecomposable injective modules in $\mathcal{F}$. Let $M = \bigoplus_{j=1}^n I_j$. Then* $\mathrm{End}_R(M)$ *is semisimple artinian if and only if $\mathrm{S}_\tau(I_j) = I_j$ for all $j \leqq n$.*

PROOF. Combine Corollary 4.2, Theorem 4.5, Corollary 4.4, and Theorem 4.6.

Let R have DCC on τ-closed left ideals, and let A be a uniform quasi-injective module in $\mathcal{F}$. A bound for the index of nilpotency of $N = \{f \in \mathrm{End}_R(A) \mid \ker f \text{ is essential in } A\}$ is given in [11, Prop. 2.3] whenever $A \in \mathcal{F}$ and A has the ascending chain condition of τ-closed submodules. Our next result provides an alternative bound in terms of linkage without using the ascending chain condition on τ-closed submodules of A.

THEOREM 4.8. *Let R have DCC on τ-closed left ideals, let A be a uniform quasi-injective module in $\mathcal{F}$, let D be the minimal τ-primitive ideal associated with $E(A)$, and let $N = \{f \in \mathrm{End}_R(A) \mid \ker f$ is essential in $A\}$. Then the index of nilpotency of N is less than or equal to the number of layers to which D is linked.*

PROOF. The proof is by induction on the number k of layers to which D is linked.

Case $k = 1$. D is linked to the first layer of A by definition. Since $k = 1$, $\mathrm{End}_R(E(A))$ is a division ring; so $J(\mathrm{End}_R(E(A))) = 0$. But any $f : A \to A$ can be extended to $g : E(A) \to E(A)$. If $f \in N$, then $g \in J(\mathrm{End}_R(E(A))) = 0$; so $f = 0$. Hence $N = 0$.

Assume that the result is true whenever D is linked to less than k layers of a quasi-injective uniform module in $\mathcal{F}$. Let D be linked to exactly k layers of A. Let the highest layer of A to which D is linked be the q-th layer. Then $\mathrm{Sc}_\tau^{q-1}(A)$ is a uniform quasi-injective module in $\mathcal{F}$ such that D is linked to exactly $k-1$ layers of $\mathrm{Sc}_\tau^{q-1}(A)$. By the induction hypothesis, if

$$N' = \{f \in \mathrm{End}_R(\mathrm{Sc}_\tau^{q-1}(A)) \mid \ker f \text{ is essential}\},$$

then $(N')^{k-1} = 0$.

Let $f_1, f_2, \ldots, f_k \in N$. Then the restriction of each f_i to $\mathrm{Sc}_\tau^{q-1}(A)$ is in N'. Thus $\ker f_2 f_3 \cdots f_k \supseteq \mathrm{Sc}_\tau^{q-1}(A)$. Hence $f_2 f_3 \cdots f_k$ induces a homomorphism from $\mathrm{Sc}_\tau^{q}(A)/\mathrm{S}_\tau^{q-1}(A) \to A$. By Proposition 1.9 we obtain $f_2 f_3 \cdots f_k(\mathrm{Sc}_\tau^q(A)) \subseteq \mathrm{Sc}_\tau(A) = \mathrm{Cl}_\tau \mathrm{S}_\tau(A)$. Since $\ker f_1$ is essential in A, then $\ker f_1 \cap C \neq 0$ for each nonzero τ-cocritical submodule C of A. Since $A \in \mathcal{F}$, it follows that $f_1(\mathrm{S}_\tau(A)) = 0$ and hence that $f_1(\mathrm{Sc}_\tau^1(A)) = f_1(\mathrm{Cl}_\tau \mathrm{S}_\tau(A)) = 0$. Therefore, $f_1 f_2 \cdots f_k(\mathrm{Sc}_\tau^q(A)) = 0$.

We claim that $f_1 f_2 \cdots f_k = 0$. For if not, then there exists an integer $m \geqq q$ such that $f_1 f_2 \cdots f_k(\mathrm{Sc}_\tau^m(A)) = 0$, but $f_1 f_2 \cdots f_k(\mathrm{Sc}_\tau^{m+1}(A)) \neq 0$. Then $f_1 f_2 \cdots f_k$ induces a nonzero homomorphism: $\mathrm{Sc}_\tau^{m+1}(A)/\mathrm{Sc}_\tau^m(A) \to A \subseteq E(A)$. By Lemma 4.3, D is linked to the $(m+1)$-th layer of A, which contradicts our choice of q. Thus $f_1 f_2 \cdots f_k = 0$ as desired.

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